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On a fixed point theorem of mixed monotone operators and applications

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Abstract. In this paper, by introducing $\tau-\varphi-$ mixed monotone operators in ordered Banach spaces, the existence and uniqueness theorems for the operators are obtained. In the end, the new results are used to study the existence of positive solutions to second order equations with *Neumann* boundary conditions.

Key words Mixed monotone operator; Fixed point theorem; Cone; Neumann boundary conditions

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1 Introduction and preliminaries

Mixed monotone operators are introduced by Dajun Guo and V. Lakshmikantham in [1] in 1987. Thereafter many authors have investigated these kinds of operators in Banach spaces and obtained a lot of interesting and important results $[2-7]$. These results are used extensively in nonlinear differential and integral equations. But in these results, continuity and compactness are the two basal conditions. These results need to restrict the operators to be completely continuous, weakly compact or having lower and upper solutions and so on.

In this paper, by introducing $\tau-\varphi-$ mixed monotone operators in ordered Banach spaces, we modify the methods in $[2,6]$ to obtain some new existence and uniqueness theorems of fixed points for $\tau-\varphi-$ mixed monotone operators. Finally we apply the main result in this paper to study the existence of positive solutions to second order equations with Neumann boundary conditions.

Let E be the Banach space, and θ be a zero element of E. Let P be a cone of E. We define a partial ordering $\leq$ with respect to P by $x \leq y$ if only if $y-x \in P$. A cone $P \subset E$ is called normal if there is a number $N>0$, such that $\theta \leq x \leq y$, implies $\| x \|\leq N \| y \|, \forall x, y \in E$. The least positive number N satisfying the above inequality is called the normal constant of P. Putting $P^{\circ}=\{x \in P : x$ is an interior point of $P\}$, a cone P is said to be solid if its interior $\mathring{P}$ is nonempty. If $x \leq y$ and $x \neq y$, we write $x<y$; if cone P is a solid and $y-x \in P^{\circ}$, we write $x \ll y$. $A: P \times P \longrightarrow P$ is said

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to be a mixed monotone operator if $A(x,y)$ is increasing in x and decreasing in y, i.e., $x_i, y_i (i=1,2) \in P,\ x_1 \leq x_2,\ y_1 \geq y_2$, implies $A(x_1,y_1) \leq A(x_2,y_2)$.

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$, such that $\lambda x \leq y \leq \mu x$. Clearly, $x \sim y$ is an equivalence relation. Given $h > \theta$, we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$ is convex for all $\lambda > 0$. If $\mathring{P} \neq \phi$ and $h \in \mathring{P}$, it is clear that $P_h = \mathring{P}$.

Lemma 2.1[8] Let P be a normal cone in a real Banach space E. Suppose that $\{x_n\}$ is a monotone sequence which has a subsequence $\{x_{n_i}\}$ which converges to a x^*, then $\{x_n\}$ also converges to x^*, moreover, if $\{x_n\}$ is a monotone increasing sequence, then $x_n \leq x^*$ $(n = 1,2,3,\cdots)$; if $\{x_n\}$ is a monotone decreasing sequence, then $x_n \geq x^*$ $(n = 1,2,3,\cdots)$.

2 Main Results

In this section, we present our main results. To begin with, we give the definition of $\tau-\varphi-$ mixed monotone operator.

Definition 2.1 Let P be a normal cone in a real Banach space E and $A : P \times P \longrightarrow P$ is a mixed monotone operator. We say that A is a $\tau - \varphi-$ mixed monotone operator if there exist two positive-valued functions $\tau(t), \varphi(t)$ on interval (a,b) such that

(**i**) $\tau : (a,b) \longrightarrow (0,1)$ is a surjection;

(**ii**) $\varphi(t) > \tau(t), \forall t \in (a,b)$;

(**iii**) $A(\tau(t)x, \frac{1}{\tau(t)}y) \geq \varphi(t)A(x,y), \forall t \in (a,b), \forall x,y \in P$.

Then we say A is a $\tau - \varphi-$ mixed monotone operator.

Theorem 2.1 Let P be a normal cone in a real Banach space E, and let $A : P \times P \longrightarrow P$ be a $\tau - \varphi-$ mixed monotone operator. In addition, suppose that there exists $h \in P \setminus \theta$ such that $A(h,h) \in P_h$.Then

($\mathbf{H}_1$) there exist $u_0, v_0 \in P_h$and $r \in (0,1)$ such that $rv_0 \leq u_0 < v_0, u_0 \leq A(u_0,v_0) \leq A(v_0,u_0) \leq v_0$;

($\mathbf{H}_2$) operator A has a unique fixed point x^* in $[u_0, v_0]$;

($\mathbf{H}_3$) for any initial $x_0, y_0 \in P_h$, constructing successively the sequence $x_n = A(x_{n-1}, y_{n-1}), y_n = A(y_{n-1}, x_{n-1}), n = 1,2,3\cdots$, we have $\| x_n - x^* \| \longrightarrow 0$, and $\| y_n - x^* \| \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof (1) Proof of ($\mathbf{H}_1$). Since $A(h,h) \in P_h$, we can choose a sufficiently small number $e_0 \in (0,1)$ such that

$$e_0 h \leq A(h,h) \leq \frac{1}{e_0}h.$$

It follows from (**i**) that there exists $t_0 \in (a,b)$ such that $\tau(t_0) = e_0$, and hence

$$\tau(t_0)h \leq A(h,h) \leq \frac{1}{\tau(t_0)}h. \tag{2.1}$$

By (**ii**), we know that $\frac{\varphi(t_0)}{\tau(t_0)} > 1$. So we can take a sufficiently large positive integer k such that

$$(\frac{\varphi(t_0)}{\tau(t_0)})^k \geq \frac{1}{\tau(t_0)}. \tag{2.2}$$

It is clear that

$$(\frac{\tau(t_0)}{\varphi(t_0)})^k \leq \tau(t_0). \tag{2.3}$$

Put $u_0 = (\tau(t_0))^k h, v_0 = \frac{1}{(\tau(t_0))^k}h$. Evidently, $u_0, v_0 \in P_h$ and $u_0 = (\tau(t_0))^{2k} v_0 < v_0$. Take any $r \in (0, (\tau(t_0))^{2k})$, then $r \in (0, 1)$ and $u_0 \geq r v_0$. Construct successively the sequences

$$u_n = A(u_{n-1}, v_{n-1}), v_n = A(v_{n-1}, u_{n-1}), n = 1, 2, 3 \cdots .$$

By using the mixed monotone properties of operator A, we have

$$u_1 = A(u_0, v_0) \leq A(v_0, u_0) = v_1.$$

Likewise,

$$u_2 = A(u_1, v_1) \leq A(v_1, u_1) = v_2.$$

In a general way, $u_n \leq v_n, n = 1, 2, 3 \cdots$.

Furthermore, by using the condition (2.1) and noticing inequality (2.2), we have

$$\begin{aligned} u_1 =& A(u_0, v_0) = A((\tau(t_0))^k h, \frac{1}{(\tau(t_0))^k} h) \\ \geq& \varphi(t_0) A((\tau(t_0))^{k-1} h, \frac{1}{(\tau(t_0))^{k-1}} h) \\ \geq& \varphi^2(t_0) A((\tau(t_0))^{k-2} h, \frac{1}{(\tau(t_0))^{k-2}} h) \\ \geq& \cdots \geq \varphi^k(t_0) A((h, h) \geq \varphi^k(t_0)\tau(t_0) h \geq \tau^k(t_0) h = u_0. \end{aligned}$$

From condition (**iii**), we have

$$A(x, y) = A(\tau(t)\frac{1}{\tau(t)}x, \frac{1}{\tau(t)}\tau(t)y) \geq \varphi(t) A(\frac{1}{\tau(t)}x, \tau(t)y), \forall x, y \in P, a < t < b.$$

and hence

$$A(\frac{1}{\tau(t)}x, \tau(t)y) \leq \frac{1}{\varphi(t)} A(x, y), \forall x, y \in P, a < t < b.$$

Thus we have

$$\begin{aligned} v_1 =& A(v_0, u_0) = A(\frac{1}{(\tau(t_0))^k} h, (\tau(t_0))^k h) \\ \leq& \frac{1}{\varphi(t)} A((\tau(t_0))^{-(k-1)} h, (\tau(t_0))^{k-1} h) \\ \leq& \frac{1}{\varphi^2(t_0)} A((\tau(t_0))^{-(k-2)} h, (\tau(t_0))^{k-2} h) \\ \leq& \cdots \leq \frac{1}{\varphi^k(t_0)} A((h, h) \leq \frac{1}{\varphi^k(t_0)} \frac{1}{\tau(t_0)} h. \end{aligned}$$

An application of (2.3) yields

$$A(v_0, u_0) \leq \frac{1}{\tau(t_0)^k} h = v_0.$$

Thus we have

$$u_0 \leq u_1 \leq v_1 \leq v_0.$$

The proof of $(\mathbf{H}_1)$ is complete.

(2) Proof of $(\mathbf{H}_2)$. By induction, it is easy to get that

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0. \tag{2.4}$$

Let

$$r_n = \sup\{r > 0 \mid u_n \ge r v_n\}, n = 1, 2, 3 \cdots .$$

Thus we have $u_n \ge r_n v_n, n = 1, 2, 3 \cdots$, and then

$$u_{n+1} \ge u_n \ge r_n v_n \ge r_n v_{n+1}, n = 1, 2, 3 \cdots .$$

Thus we have $r_{n+1} \ge r_n, i.e, r_n$ is increasing with $r_n \subset (0, 1]$. Suppose $r_n \longrightarrow r^*$ as $n \longrightarrow \infty$. Then $r^* = 1$. Indeed, suppose to the contrary that $0 < r^* < 1$. By (**i**), there exist $t_1 \in (a, b)$ such that $\tau(t_1) = r^*$. We distinguish two cases:

Case one: There exists an integer N such that $r_N = r^*$. In this case we know $r_N = r^*$ for all $n \ge N$. So for $n \ge N$, we have

$$u_{n+1} = A(u_n, v_n) \ge A(r^* v_n, \frac{1}{r^*} u_n) = A(\tau(t_1) v_n, \frac{1}{\tau(t_1)} u_n) \ge \varphi(t_1) A(v_n, u_n) = \varphi(t_1) v_{n+1}.$$

By the definition of r_n, we get $r_{n+1} = r^* \ge \varphi(t_1) > \tau(t_1) = r^*$. This is a contradiction.

Case two: For all integer $n, r_n < r^*$.Then we obtain $0 < \frac{r_n}{r^*} < 1$. By (**i**), there exist $s_n \in (a, b)$ such that $\tau(s_n) = \frac{r_n}{r^*}$. So we have

$$\begin{aligned} u_{n+1} =& A(u_n, v_n) \ge A(r_n v_n, \frac{1}{r_n} u_n) \\ =& A(\tau(s_n) r^* v_n, \frac{1}{\tau(s_n) r^*} u_n) \ge \varphi(s_n) A(r^* v_n, \frac{1}{r^*} u_n) \\ =& \varphi(s_n) A(\tau(t_1) v_n, \frac{1}{\tau(t_1)} u_n) \ge \varphi(s_n)\varphi(t_1) A(v_n, u_n) \\ =& \varphi(s_n)\varphi(t_1) v_{n+1}. \end{aligned}$$

By the definition of r_n, we have

$$r_{n+1} \ge \varphi(s_n)\varphi(t_1) > \tau(s_n)\varphi(t_1) = \frac{r_n}{r^*}\varphi(t_1).$$

Let $n \longrightarrow \infty$, we get $r^* \ge \frac{r^*}{r^*}\varphi(t_1) > \tau(t_1) = r^*$, which also is a contradiction. Thus $r^* = 1$. For any natural number p we have

$$\theta \le u_{n+p} - u_n \le v_n - u_n \le v_n - r_n v_n = (1 - r_n) v_n \le (1 - r_n) v_0,$$

$$\theta \le v_n - v_{n+p} \le v_n - u_n \le v_n - r_n v_n = (1 - r_n) v_n \le (1 - r_n) v_0.$$

Since P is normal, we have

$$\| u_{n+p} - u_n \| \le N \| (1 - r_n) v_0 \| \to 0 (n \to \infty),$$

$$\| v_{n+p} - v_n \| \le N \| (1 - r_n) v_0 \| \to 0 (n \to \infty).$$

Here N is the normality constant.

So $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Because E is complete, by *Lemma* 2.1, there exist $u^*, v^* \in [u_0, v_0]$ such that $u_n \to u^*$, $v_n \to v^*$ as $n \to \infty$. By (2.4), we know that $u_n \le u^* \le v^* \le v_n$ and

$$\theta \le v^* - u^* \le v_n - u_n \le (1 - r_n) v_0.$$

Further

$$\| v^* - u^* \| \leq N \| (1-r_n)v_0 \| \to 0 (n \to \infty),$$

we know that $u^* = v^*$. Let $x^* := u^* = v^*$, we obtain

$$u_{n+1} = A(u_n, v_n) \leq A(x^*, x^*) \leq A(v_n, u_n) = v_{n+1}.$$

Let $n \to \infty$ and we get $x^* = A(x^*, x^*)$. That is, A has a fixed point x^*in $[u_0, v_0]$.

In the following, we prove that x^* is the unique fixed point of A in P_h. In fact, suppose $\overline{x}$ is another fixed point of A in P_h. Since $x^*, \overline{x} \in P_h$, there exist positive numbers $\mu_1, \mu_2, \lambda_1, \lambda_2 > 0$ such that

$$\mu_1 h \leq x^* \leq \lambda_1 h, \mu_2 h \leq \overline{x} \leq \lambda_2 h.$$

Then we obtain

$$\overline{x} \geq \mu_2 h = \frac{\mu_2}{\lambda_1}\lambda_1 h \geq \frac{\mu_2}{\lambda_1} x^*.$$

Let $e_1 = \sup\{e > 0 \mid \overline{x} \geq e x^*\}, n = 1, 2, 3 \cdots$. Evidently, $0 < e_1 < \infty$. Furthermore, we can prove $e_1 \geq 1$. If $0 < e_1 < 1$. From (**i**), there exists $t_2 \in (a, b)$ such that $\tau(t_2) = e_1$. Then

$$\overline{x} = A(\overline{x}, \overline{x}) \geq A(e_1 x^*, \frac{1}{e_1} x^*) = A(\tau(t_2)x^*, \frac{1}{\tau(t_2)} x^*) \geq \varphi(t_2) A(x^*, x^*) = \varphi(t_2) x^*.$$

Since $\varphi(t_2) > \tau(t_2) = e_1$, this contradicts the definition of e_1. Hence $e_1 \geq 1$, and then we get $\overline{x} \geq e_1 x^* \geq x^*$. Similarly we can prove $x^* \geq \overline{x}$; thus $x^* = \overline{x}$. Therefore, A has a unique fixed point x^*in P_h. Note that$[u_0, v_0] \subset P_h$, so A has a unique fixed point x^* in $[u_0, v_0]$.

(3) Proof of $(\mathbf{H}_3)$. For any initial $x_0, y_0 \in P_h$, we can choose a small number $e_2 \in (0, 1)$ such that

$$e_2 h \leq A(h, h) \leq \frac{1}{e_2} h.$$

It also follows from (**1**) that there exists $t_3 \in (a, b)$ such that $\tau(t_3) = e_2$, and hence

$$\tau(t_3) h \leq A(h, h) \leq \frac{1}{\tau(t_3)} h.$$

we can take a positive integer m such that

$$(\frac{\varphi(t_3)}{\tau(t_3)})^m \geq \frac{1}{\tau(t_3)}.$$

Put $\overline{u}_0 = (\tau(t_3))^m h, \overline{v}_0 = \frac{1}{(\tau(t_3))^m} h$. Evidently, $\overline{u}_0, \overline{v}_0 \in P_h$ and $\overline{u}_0 < x_0 < \overline{v}_0$. Let

$$\overline{u}_n = A(\overline{u}_{n-1}, \overline{v}_{n-1}), \overline{v}_n = A(\overline{v}_{n-1}, \overline{u}_{n-1}), n = 1, 2, 3 \cdots .$$

By using the mixed monotone properties of operator A , we get

$$\overline{u}_n \leq x_n \leq \overline{v}_n, \overline{u}_n \leq y_n \leq \overline{v}_n, n = 1, 2, 3 \cdots .$$

Taking into account that P is normal, we immediately conclude that

$$\lim_{n \to +\infty} x_n = \lim_{n \to +\infty} y_n = x^*.$$

The proof of *Theorem* 2.1 is complete.

Remark 2.1: Let P be a solid cone in a real Banach space E. If we suppose that operator $A : P^\circ \times P^\circ \longrightarrow P^\circ$, then $A(h,h) \in P_h$ is automatically satisfied.

This proves the following corollary.

Corollary 2.1 Let P be a solid cone in a real Banach space E. Suppose $A : P^\circ \times P^\circ \longrightarrow P^\circ$ be a $\tau - \varphi-$ mixed monotone operator.Then A has a unique fixed point x^* in P°.

Remark 2.2: When $\tau(t) = t, t \in (0,1)$ and $\varphi(t) = t(1+v(t))$ or $t^{\alpha(t)}$ with $\alpha(t) \in (0,1), v(t) > 0$ for $t \in (0,1)$. Theorem 2.1, Corollary 2.1 also hold. The corresponding results in [2,3,4,6] turn out to be special cases of our main results, see [2, Theorem 2.1].

3 Applications

In this section, we use Theorem 2.1 to study the existence of positive solutions to second order singular equations with *Neumann* boundary conditions.

$$\begin{cases} u''(t) + \beta^2 u(t) = f(t, u(t), u(t)), 0 < t < 1, \\ u'(0) = u'(1) = 0. \end{cases} \tag{3.1}$$

Here $\beta \in (0, \frac{\pi}{2})$ is a constant and the nonlinearity $f : C([0,1] \times [0,+\infty) \times [0,+\infty), [0,+\infty))$.

Let $E = C[0,1]$ be a *Banach* space with maximum norm $\| \cdot \|$, $P = \{u \in E \mid u(t) \geq 0, \forall t \in [0,1]\}$, then P is a normal solid cone in Banach space E.

Define an operator $A : P \times P \longrightarrow E$

$$A(u,v) = \int_0^1 G(t,s) f(s, u(s), v(s)) ds, \ t \in [0,1].$$

Where

$$G(t,s) = \begin{cases} \frac{\cos\beta s \cos\beta(1-t)}{\beta\cos\beta}, 0 \leq s \leq t \leq 1, \\ \frac{\cos\beta t \cos\beta(1-s)}{\beta \sin\beta}, 0 \leq t \leq s \leq 1. \end{cases}$$

It is to see that $G(t,s) > 0$ for all $m \in (0, \frac{\pi}{2})$ and $\int_0^1 G(t,s))ds = \frac{1}{\beta^2}$.

Theorem 3.1 *Assume that*

(H_1) *$f(t,u,v)$ is increasing in u for fixed (t,v); $f$is decreasing in u for fixed (t,u).*

(H_2) *there exist two positive-valued functions $\tau(t), \varphi(t)$ on interval $(0,1)$ such that $\tau : (a,b) \longrightarrow (0,1)$ is a surjection, $\varphi(t) > \tau(t), \forall t \in (0,1)$ which satisfy*

$$f(t, \tau(\lambda)u, \frac{1}{\tau(\lambda)} v) \geq \varphi(\lambda) f(t,u,v), \forall t, \lambda \in (0,1), u, v \in P.$$

(H_3) *there exist two constants $M_1, M_2 > 0$ and $h \in P \setminus 0$ such that*

$$M_1 h(t) \leq f(t, h(t), h(t)) ds \leq M_2 h(t), \ t \in (0,1).$$

Then equation (3.1) has a unique positive solution. Moreover,for any initial $x_0, y_0 \in P_h$, constructing successively the sequence $x_n = A(x_{n-1}, y_{n-1}), y_n = A(y_{n-1}, x_{n-1}), n = 1,2,3\cdots$, we have $\| x_n - x^* \| \longrightarrow 0$, and $\| y_n - x^* \| \longrightarrow 0 as\ n \longrightarrow \infty$.

Proof It is easy to see that u is a solution of problem (3.1) if and only if $u = A(u,u)$. Note that since $f(t,u(t),u(t)) \geq 0$, we have $A(t,u(t),u(t)) \geq 0$ for $t \in (0,1)$. It is obviously to know that $A : P \times P \longrightarrow P$ be a mixed monotone operator. For any $u, v \in P, 0 < t < 1$, by the condition (H_2), we have

$$A(\tau(\lambda)u, \frac{1}{\tau(\lambda)}v) \geq \varphi(\lambda)A(u,v),\ \forall u,v \in P, 0 < t < 1.$$

So operator$A$ is a $\tau - \varphi-$ mixed monotone operator. Further, from (H_3) we know that there exist two constants $M_1, M_2 > 0$ and $h \in P \setminus 0$ such that

$$M_1 h(t) \leq \int_0^1 G(t,s)f(s,h(s),h(s))ds \leq M_2 h(t),\ t \in (0,1).$$

So the conditions of $Theorem$ 2.1 hold, $Theorem$ 3.1 is proved.

Remark 3.1: There exist many functions which satisfy the conditions of $Theorem$ 3.1.

Example 3.1: We give an example to illustrate Theorem 3.1. Consider the following Neumann boundary problems

$$\begin{cases} u''(t) + \beta^2 u(t) = \sqrt{1+u(t)} + \frac{1}{\sqrt[4]{u(t)}}, 0 < t < 1, \\ u'(0) = u'(1) = 0. \end{cases} \tag{3.2}$$

It is easy to show that $Neumann$ boundary problems satisfy the conditions of Theorem 3.1. So equation (3.2) has a unique positive solution.

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